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ON THE FLOW OVER A CONE-CYLINDER BODY AT MACH NUMBER ONE

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WRIGHT AIR DEVELOPMENT CENTER

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ON THE FLOW OVER A CONE-CYLINDER BODY AT MACH NUMBER ONE

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November 1952

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FOREWORD

This report was prepared by the Wind Tunnel Branch, Aircraft Laboratory, Aeronautics Division, Wright Air Development Center, Wright-Patterson Air Force Base, Dayton, Ohio. The project was administered under Research Development Order Number 458-429, Two Dimensional and Axial-Symmetric Transonic Flows. The author of this report was also the project engineer.

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ABSTRACT

The flow over a cone-cylinder body at Mach number one and zero angle of attack is computed by a numerical method in which the subsonic region is computed by the relaxation method while the supersonic region is constructed by the method of characteristics. The sonic line is then determined by an iterative process.

The results indicate that transonic similarity law proposed by von Karman and Oswatitsch and Berndt based upon slender body theory can be extended to cover cone-cylinder bodies of practical slenderness ratios.

The security classification of the title of this report is UNCLASSIFIED.

PUBLICATION REVIEW

This report has been reviewed and approved.

FOR THE COMMANDING GENERAL:

D. D. MCKEE
Colonel, USAF
Acting Chief, Aircraft Laboratory
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INTRODUCTION

In the present paper the flow pattern over a cone-cylinder body
(circular conical nose with a cylindrical afterbody) at Mach number
one and at zero angle of attack is computed. This problem offers con-
siderable difficulty since the flow differential equation remains
non-linear even with simplifications and transformations which were so
successful for the linearization in the planar case. It is not diffi-
cult to formulate the boundary value problem, but there is little to
base the correctness of the formulation so as to insure the existence
of an unique solution. The absence of existence and uniqueness proofs
for problems of the present nature is due mainly to the non-linearity
of the differential equation, but the mixed elliptic (subsonic) and
hyperbolic (supersonic) character of the equation, as well as the nature
of the boundary conditions, are also responsible.

In the following, we shall, therefore, formulate the flow problem
in the physical plane and seek a solution numerically assuming that it
exists, and is unique. Whether this is actually the case, however,
cannot be decided by a numerical computation.

Two general and successful numerical methods to solve boundary
value problems are the variational procedure exemplified by the Rayleigh-
Ritz method, and the finite difference method utilizing the relaxation
process. These methods are well established for boundary value pro-
blems of an elliptic type, such as in potential theory, but the co-
existence of the supersonic region with the subsonic region makes their
use questionable in the case of transonic problems.

The variational method is based on the equivalence of solving a
boundary value problem with the determination of a stationary value
(an extremum in the case of problems in potential theory) of a certain
integral expression extending over the region of the problem. Neces-
sary conditions for the existence of the stationary value implied that
either the value of the sought function or the so-called "natural
boundary conditions" must be prescribed over the entire closed boundary
of the problem. This is, of course, compatible with the problems of
the elliptic type, but difficulty will arise in the case of mixed
elliptic-hyperbolic type of problems as the present one where it is
well-known that in order to obtain a physically possible solution, a
gap in the supersonic region must be left open along which no condition
can be prescribed. Thus, for transonic type of boundary value prob-
lems in which either the sought function or the natural boundary
condition is prescribed there is no possibility to use the variational
method. The situation for problems with other boundary conditions is
considerably more obscure. No real basis exists for these cases on
which to attempt such procedures as the Rayleigh-Ritz method as in
potential theory. The fact that the stationary value of the integral
expression corresponding to the present transonic problem cannot be

an extremum as in the case of potential problems warrants some caution. It, therefore, appears considerably more reliable to treat directly the boundary value problem by a numerical method.

In the finite difference method for the numerical solution of the problem, instead of seeking the value of the solution at every point of the domain of the problem, only the value at a finite number of points, at the so-called lattice points, are sought. Here partial derivatives are replaced by partial difference expressions, and the boundary value problem reduces to solving a system of algebraic equations, the number of equations being equal to the number of lattice points selected. The transition from the approximate solutions found in this way to the exact solution is obtained by systematically increasing the number of lattice points and finding the limit of the sequence of these approximate solutions as the number of lattice points is increased. In this process the arrangement of the lattice points plays an important role. According to the results of Courant, Friedrichs, and Lewy, (Reference 1) it would follow that in the hyperbolic region with the present transonic differential equation, the lattice points must be so arranged that the "domain of dependence" for a given set of initial values in the case of the difference equation must at least fall within the "domain of dependence" for the same initial values for the differential equation in order to obtain the correct limit, if it exists at all. (In other words for a rectangular system of lattice points the ratio of the distances between points in the y and x directions must be less than the local slope of the characteristics.) In the present problem, we cannot preselect a satisfactory arrangement of the lattice points in the supersonic region adjacent to the sonic line since the characteristics which determine the "domain of dependence" are extremely steep; moreover the location of the characteristics is known only after the problem has been solved. The question is still left open whether one can obtain a reasonable approximation with a lattice point arrangement other than the required one since one seldom takes the limit to an infinitely small distance between the points but retains a finite number of lattice points. This question could of course be answered if the behavior of the truncation errors is known in its dependence upon the arrangement of the lattice points.

In order to avoid the above difficulties, we shall separate the supersonic region from the subsonic and then apply to each region a numerical technique which is known to be reliable. First the position of the sonic line which separates the two regions will be estimated. The subsonic region is then computed by the relaxation method, while in the supersonic region the method of characteristics is used. A method of successive approximation is then introduced to find the correct location of the sonic line so as to provide a compatible subsonic and supersonic solution.

SECTION I

THE FLOW OVER A CONE-CYLINDER BODY

The flow pattern in a meridian plane for the cone-cylinder body at Mach number one and zero angle of attack can be expected to be qualitatively the same as for a two-dimensional wedge. The general features of the flow in the latter case are well known, so that we shall repeat here only the salient points.

It is known that at infinity one has the sonic free stream conditions. The nose of the body, at Point B of Figure 1, is a stagnation point; this means that the usual transonic perturbation method, which we shall use in the following, will not be valid in the neighborhood of this point. At the shoulder at Point C one has the starting point of the sonic line which extends from there laterally to infinity. This line separates the flow into the elliptic region upstream of this line and the hyperbolic region downstream, but unfortunately its location is not known in advance. In the neighborhood of the shoulder one has a Meyer expansion fan from which Mach waves travel into the flow. Some of these waves will reach the sonic line while others occurring later in the fan will not. There is one wave, the limiting Mach wave (CE in Figure 1), which separates these two classes. This wave forms the downstream limit of that part of the supersonic region which will influence the upstream subsonic flow. This is clear since any small disturbance originating downstream of the limiting Mach wave will propagate along a Mach wave which never reaches the sonic line and will, therefore, not influence the subsonic region. The position of the limiting Mach wave, just as the position of the sonic line, is unknown. For the determination of the subsonic portion of the flow pattern it is then necessary to consider only the flow upstream of the limiting Mach wave. The remainder of the flow downstream is formulated subsequently as a purely hyperbolic problem and computed by standard methods.

SECTION II

THE BASIC DIFFERENTIAL EQUATION

Let us first define the following symbols:

x and y are the Cartesian coordinates in a meridian plane with

the x-axis coinciding with the axis of symmetry,

$\bar{\phi}$ the velocity potential,

a^* the critical velocity,

ϕ the perturbation potential defined by $\phi a^* = \bar{\phi} - a^* x$, and

k   the ratio of specific heats.

With the approximations usually made in transonic theory (see References 2 or 4 for example) the basic flow equation for the velocity potential may be simplified to

$$-(K+1)\phi_x \phi_{xx} + \phi_{yy} + \frac{1}{y}\phi_y = 0 \tag{1}$$

Here the difference as compared to the planar case is the appearance

of the term $\frac{1}{y}\phi_y$. The presence of this term makes it impossible to

linearize this equation as in the planar case by a hodograph transformation. The attempt to use other more general transformations (see Reference 3) was also unsuccessful.

Equation (1) for the perturbed potential is of the elliptic type for $\phi_x < 0$ and hyperbolic when $\phi_x > 0$. For the hyperbolic region the equation of the characteristics and the compatibility conditions required along the characteristics are given by

$$\frac{dy}{dx} = \pm (K+1)^{-\frac{1}{2}} \phi_x^{-\frac{1}{2}} \tag{2a}$$

and

$$\pm \frac{2}{3}(K+1)^{\frac{1}{2}} \frac{d(\phi_x^{\frac{3}{2}})}{dy} - \frac{d(\phi_y)}{dy} = \frac{1}{y}\phi_y \tag{2b}$$

Here the signs must be taken consistently in the two equations.

With the transformation

$$x = \xi$$
$$y = (K+1)^{-\frac{1}{2}} e^{\eta} \tag{3}$$
$$\phi(x,y) = \varphi(\xi,\eta)$$

Equation (1) becomes

$$-\varphi_\xi \varphi_{\xi\xi} + e^{-2\eta} \varphi_{\eta\eta} = 0 \tag{4}$$

The reason for the above transformation is to provide a coordinate system which would stretch the region near the body where one would expect the greatest change within the flow. The computations for the present problem showed that it was indeed the "natural" coordinate system for the vicinity of the body.

The corresponding equation of the characteristics and the compatibility condition for this equation are

$$\frac{d\eta}{d\xi} = \pm e^{-\eta} \varphi_\xi^{-\frac{1}{2}}$$

(5a)

and

$$\frac{d(\varphi_\xi)}{d(\varphi_\eta)} = \frac{d\eta}{d\xi}$$

(5b)

If we introduce a "hodograph plane" with coordinates φ_η and φ_ξ, then it is seen from Equation (5b) that the slopes of the characteristics in the ξ, η-plane and in the "hodograph plane" are the same at corresponding points. Using this fact a simple graphical procedure can be used in which one can avoid measuring Mach wave segments as is necessary in the usual step by step construction using Equations (2a) and (2b). On the other hand Equation (5a) for the Mach wave slopes is expressed in a less convenient form than in Equation (2a) because of the presence of both physical and hodograph variables in the right hand term.

SECTION III

THE BOUNDARY CONDITIONS

The boundary of the problem is first shown in Figure 1 by the contour ABCEDA, and it consists of the axis of symmetry, the cone surface, and the limiting Mach wave; otherwise the flow extends to infinity.

The boundary condition required along the axis of symmetry upstream of the cone and along the cone is that the velocity normal to this boundary must vanish. This means that along the axis of symmetry one has $\phi_y = 0$ and that along the surface of the cone $\phi_y = \theta_0$ where θ_0 is the semi-nose angle of the cone. In the neighborhood of the shoulder one has a Meyer expansion; that is, a family of Mach waves characterized by the plus sign in Equations (2a) and (2b) starts at the shoulder with the slopes $\frac{dy}{dx} = (\kappa+1)^{-\frac{1}{2}} \phi_x^{-\frac{1}{2}}$ where the value of ϕ_x at the shoulder for a particular wave of the fan is related to ϕ_y by the equation

$$-\frac{2}{3}(\kappa+1)^{\frac{1}{2}} \phi_x^{\frac{3}{2}} + \phi_y = \theta_0$$

(6)

Finally we require that the flow tends to the sonic free stream conditions as we go to infinity.

Along the limiting Mach wave no boundary condition may be prescribed. This is physically clear since the knowledge of the subsonic solution together with the supersonic boundary condition at the

shoulder will completely determine the flow conditions along the limiting Mach wave. This gap in the supersonic boundary is quite typical of transonic boundary value problems.

For practical computations the boundary condition at infinity must be replaced by more suitable conditions along boundaries at a finite distance from the body. For this purpose we use the solution computed in Reference 4 which gives the asymptotic behavior of the flow at a large distance from an axial symmetric body; we then stipulate that the flow must assume this asymptotic solution on a new boundary consisting of a line of constant x and a line of constant y sufficiently remote from the body that this solution is valid. (Strictly speaking, the asymptotic solution will agree with the sought solution only in the limit that one approaches infinity in the physical plane; at any finite distance from the body there will be a discrepancy between the two. However we shall be concerned only with approximate solutions so that as long as the differences of the exact solution of the problem from both the asymptotic and approximate solutions are of the same magnitude, then this discrepancy will be of no consequence.)

There is however a difficulty with the above procedure since one does not know what size body produces the flow represented by the given asymptotic solution. In other words one does not know the relative scale factor between the coordinate system of the problem and that used for the asymptotic solution. We need therefore to introduce an appropriate scale change and examine the effects of this change on the asymptotic solution. The aim of course is to find from this an alternate boundary condition which is equivalent to prescribing the asymptotic solution but which does not contain explicitly the unknown scale factor.

Let us recall first the form of the asymptotic solution. It was found by the hypothesis

$$\phi = \bar{y}^{-\frac{2}{7}} \bar{f}(\zeta) \tag{7}$$

where

$$\bar{\zeta} = (\kappa+1)^{-\frac{1}{3}} \bar{x} \, \bar{y}^{-\frac{4}{7}} \tag{8}$$

and $\bar{f}$ is given by the differential equation

$$(49\bar{f}' - 16\zeta^2)\bar{f}'' - 32\zeta\bar{f}' - 4\bar{f} = 0 \tag{9}$$

Here we denote the Cartesian coordinates in the asymptotic representation as $\bar{x}$ and $\bar{y}$ in order to distinguish them from the coordinates x and y in the actual flow. The values of f and f' as computed in Reference 4 are shown in Figure 2.

In order to examine the effect of a scale transformation upon the asymptotic solution, we first relate the coordinates in the asymptotic representation and that in the actual flow by the equations $\bar{x} = cx$ and $\bar{y} = y$. Only one scale factor c is needed here since the same change of scale in both coordinates will not change the asymptotic solution. The effect of this change of scale will now alter the scale of ζ and according to Equation (7) we must investigate the effect of this change of scale on the function f.

Consider now a scale transformation of both $\bar{f}$ and $\bar{\zeta}$ in Equation (9) such that $\bar{\zeta} = c\zeta$ and $\bar{f}(\bar{\zeta}) = c'f(\zeta)$ where c and c' are constant scale factors. If now $c' = c^3$, the differential equation will be invariant under this scale transformation. Therefore we have the result

$$\bar{f}(\bar{\zeta}) = c^3 f(\zeta) \tag{10}$$

This relation is now to be used in Equation (7) to give the effect of a scale change on the asymptotic solution.

In order to find a form of the boundary condition which is invariant with respect to the scale factor c, we seek now a linear functional relation $F(\phi_x, \phi_y, \phi, x, y) = 0$ which is fulfilled by the asymptotic representation and in which the scale factor c does not appear explicitly. A relation fulfilling these conditions is given by

$$F = 4x\phi_x + 7y\phi_y + 2\phi = 0 \tag{11}$$

There is of course still the question of uniqueness of Equation (11) which we cannot settle. We must therefore assume the validity of this condition and then check in the final solution whether or not the desired asymptotic solution is actually obtained.

With the conditions at infinity modified in this way, the downstream boundary of the problem must now be taken at the Mach wave (CFD' of Figure 3) which passes through the intersection of the sonic line and the upper boundary. The location of this Mach wave is unknown.

SECTION IV

THE BOUNDARY VALUE PROBLEM

Let us now restate our problem. We seek now a solution of the differential equation $-(K+I)\phi_x\phi_{xx} + \phi_{yy} + \frac{1}{y}\phi_y = 0$ which fulfills the the following boundary conditions:

1. Along the surface of the cone one has the condition $\phi_y = \theta_0$

where θ_0 is the half angle of the cone; and along the axis of symmetry upstream of the cone $\phi_y = 0$.

2. At the shoulder of the body, the flow is locally a Meyer expansion.

3. Sufficiently far from the body along a line of constant x and along a line of constant y one has the condition $4x\phi_x + 7y\phi_y + 2\phi = 0$ We have shown these boundary conditions in Figure 3.

SECTION V

NUMERICAL METHOD TO SOLVE THE BOUNDARY VALUE PROBLEM

It is a rather hopeless task to attempt to solve analytically the boundary value problem as it was formulated in the previous section. We must therefore carry out the solution by a suitable numerical process.

Following the procedure outlined in the introduction we first approximate the location of the sonic line (the parabolic line). On this boundary the condition $\phi_x = 0$ is prescribed. In the completely elliptic region formed in this way the finite difference method is now used.

In order to carry out this procedure it is necessary that the sought solution ϕ be regular, that is, expressible by a power series, at every point of the domain D of the problem. (In the present case, this condition is not fulfilled at the nose of the cone nor at the shoulder, but it is possible to choose boundaries arbitrarily close to the cone for which this requirement is fulfilled.) Under these conditions one can, in principle, choose within D a subdomain d_n sufficiently small such that in d_n a quadratic polynomial, for example, in the independent variables is adequate to represent the solution to a given accuracy. In such a subdomain the potential ϕ may then be expressed as

$$\phi = c_1 x^2 + c_2 y^2 + c_3 xy + c_4 x + c_5 y \tag{12}$$

where the coefficients c_i (i = 1, 2, $\cdots$, 5) are constants. These coefficients can be evaluated if one knows the value of ϕ at any five points within d_n; knowledge of ϕ at the five points will therefore be sufficient to determine it in the entire subdomain.

We can now consider the entire region D to be covered by a network of lattice points. A typical subdomain is then considered to be the region bounded by a contour enclosing a given point and its four neighboring ones.

Let us consider now a procedure to determine the values of ϕ at the lattice points. To make the discussion specific, let us first consider a system of lattice points formed by the points of intersection of lines of constant x and y taken at intervals of $\delta x = h$ and $\delta y = h'$, and examine a typical subdomain as shown in Figure 4. One first expresses the various coefficients in Equation (12) in terms of the values of ϕ at the five lattice points. The resulting expression for ϕ must now fulfill the differential equation. Inserting this expression in Equation (1) then gives

$$- \left(\frac{\phi_1 - \phi_3}{2h} \right) \left(\frac{\phi_1 + \phi_3 - 2\phi_0}{h^2} \right) + \left(\frac{\phi_4 + \phi_2 - 2\phi_0}{h'^2} \right) + \frac{1}{y_0} \left(\frac{\phi_4 - \phi_2}{2h'} \right) = 0 \quad (13)$$

where $\phi_0, \cdots \phi_4$ are the sought values of ϕ at the points as shown in Figure 4 and y_0 is the ordinate of the central lattice point. For subdomains adjacent to the boundaries one will have a slightly more complicated expression than Equation (13) due to the different arrangement of the lattice points.

The above procedure is now repeated for each interior lattice point, and in this way one will obtain as many relations, such as Equation (13), as the number of interior lattice points. These equations will be quadratic. At the boundary lattice points one will have further equations from the boundary conditions.

It is seen therefore that in the above process the boundary value problem is replaced by an algebraic problem of finding the solution of a system of quadratic equations, the number of equations being equal to the number of lattice points selected.

For the large number of points usually needed in flow computations the direct attempt to solve the system is impractical even with the assistance of large computing machines. If, however, one has an estimate of the values of ϕ then a method of iteration (the relaxation method) may be used to reduce the amount of computations. The initial guess may be obtained either from known solutions over similar bodies or from experiments; it is chosen so as to fulfill the boundary conditions, but it will generally not fulfill the difference equation everywhere. At most points there will then be a mistake (called a residual) showing the amount by which the difference equation is not fulfilled. Inasmuch as the potential equation is the condition of continuity of the flow these residuals can be interpreted as sources within the flow which must be eliminated. The essential idea of the relaxation method is now to remove these residuals one point at a time. For this purpose, one keeps the value of ϕ at all points except one fixed and then changes or relaxes the value at this point such that the difference equation (13) is fulfilled at that point. This procedure is then repeated at other points. It is to be noticed that changing

the value of ϕ at a given point will not only change the residual at the point but also that of its four neighboring points. Therefore, in the process of correcting a given point, one will disturb neighboring points which might have been previously corrected. To be useful the rate of convergence of this iterative procedure must be sufficiently rapid. For a given arrangement of the lattice points, it is useless to reduce the residuals to an order of magnitude less than that due to the truncation error. At this point one then takes a finer lattice point system and repeats the procedure until two succeeding sizes of the network give the same result to within a desirable accuracy.

There are many improvements over the basic procedure described in the previous paragraph. One useful modification which will be used in the present case is the possibility of relaxing simultaneously an entire column of lattice points, that is, all of the points along a line of constant x. The values of ϕ at neighboring points in the column are then considered to be related by the difference equation (13) which in this case will be linear; this equation together with the boundary conditions at the upper and lower boundaries then form a boundary value problem which may be solved by a standard numerical method. The obvious advantage of taking the points to be relaxed in a column is that the resulting difference equation is linear. Another equally important but less obvious reason is that relaxing a given column will not affect the residuals in the adjacent column to any extent. This is due to the fact that over most of the region, with the exception of the neighborhood of the stagnation point, the contribution of the x derivatives to the differential equation is much smaller than that due to the y derivatives.

By carrying out the relaxation method in the subsonic region one will now obtain the velocities throughout the region considered and in particular the vertical component of the velocity vector along the assumed position of the sonic line.

If now the supersonic region is computed by the method of characteristics using the location of the sonic line and the supersonic boundary condition for the construction, then one will also obtain a distribution of the vertical component of the velocity along the sonic line which will in general not agree with that obtained previously from the solution in the subsonic region. This means of course that the subsonic and supersonic regions do not have the proper continuity of the velocities at their common boundary, the sonic line. In order to correct this, one reconstructs the supersonic region such that the location of the sonic line is allowed to change so that at the new location one has the values of the vertical component of the velocity obtained from the subsonic field. The details of this construction can be seen by carrying out the method of characteristics. Changing the location of the sonic line will of course affect the solution previously computed in the subsonic region by the relaxation method. It is anticipated that this change will be confined mainly to the vicinity of the sonic line and will not extend too far upstream due to the

parabolic character of the differential equation in this region. Using the new location of the sonic line, the subsonic field will again be computed, and one will obtain another distribution of the vertical component of the velocity near the sonic line. This distribution will again be used as a basis to relocate the sonic line. This procedure is then repeated until one obtains a location of the sonic line which will yield the same distribution of the vertical velocity along the sonic line both from the relaxation method in the subsonic region and the method of characteristics in the supersonic region.

SECTION VI

DETAILS OF THE COMPUTATIONS

Using the method proposed in the previous paragraph for the transonic case we have computed the flow over a cone-cylinder body with a semi-nose angle of $\theta_0 = 1/10$. For the subsonic region we shall use the rectangular network of lattice points described previously. As the first estimation of the desired flow, the approximation found in Reference 4 for the cusped-nose body is used with modifications made near the body to take into account the difference of the body shapes. This gives also an approximate location of the sonic line. In the application of the relaxation method some difficulty is anticipated in the neighborhood of the stagnation point where the component of the velocity in the free stream direction tends to minus infinity. The nature of this singularity cannot be found in a completely satisfactory but yet simple manner. Since the perturbation differential equation is not valid in this region, we shall circumvent the difficulty by simply setting $\phi_x = -1$ at the stagnation point.

In the subsonic region the relaxation process was first carried out using the difference equation in the x,y-system, but since the values of ϕ varied so much near the body subsequent calculations were made in the ξ,η-system. See Equation (3). In either representation it was found that for a given lattice point spacing the convergence was poor. It was especially poor near the upper boundary where the boundary condition (11) was prescribed. For the region near the body with the exception of the vicinity of the nose it was rather interesting that the value of ϕ_ξ (proportional to the pressure) was nearly independent of the lattice point spacing. This would indicate that the remaining residuals would have only a small influence on the pressure distribution over the cone.

With regard to correcting the position of the sonic line, there was little trouble in determining a definite location of the sonic line for a given distribution of the vertical velocity obtained from the relaxation method. The construction of the supersonic region was first carried out using the method of characteristics in the x,y-system but due to the smallness of the ϕ_x and the rapidity with which the ϕ_y varied near the body, it was impossible to obtain reasonable

results with a practical number of lattice points. This difficulty, however, was overcome by using the method of characteristics in the ξ, η -system.

The effect of changing the position of the sonic line on the subsonic region for the present case was negligible. Due to the parabolic character of the differential equation in the vicinity of the sonic line, the abscissa coordinate was influential only as a parameter with the main effect on the solution being exerted through the boundary conditions. For those columns of lattice points which had their lower terminus on the body there was negligible influence of the sonic line position, but for those columns which occurred further downstream and which had their lowest lattice point on the sonic line, the influence was more direct. Inasmuch as the final position of the sonic line was so close to the initial estimate there was even in this case little influence due to a change of the sonic line. In any case, it was evident that the velocity distribution over the cone surface was very little influenced by the position of the sonic line.

A satisfactory match with the asymptotic solution, described in a previous section, could not be obtained at the upper boundary. This may have been due to several reasons. One is that the convergence of the solution near the upper boundary was slow, so that the correct values were not yet obtained in this region. Another possibility is that the upper boundary was not located at a sufficiently large value of y; in this case the asymptotic solution would not represent a satisfactory approximation to the correct solution.

No further attempt was made to refine the solution in this region. This was felt to be unnecessary since the flow in the vicinity of the body which is our primary concern was little influenced by the flow near the upper boundary.

The values of ϕ obtained by the computation are shown in Figure 5. It is to be noticed here that in the vicinity of the body the value of $\phi_\eta = y\phi_y$ for a given value of x is for all practical purposes a constant. This fact will be used in the next section to check the validity of the transonic similarity law proposed for axial-symmetric bodies. The distribution of ϕ_x and ϕ_y are next shown in Figures 6a and 6b. The results shown here have been made smooth near the upper boundary.

Finally in Figure 7 is shown the resulting pressure distribution over the cone. The pressure coefficient C_p consistent with the slender body theory is expressed in terms of the velocity components by the equation

$$C_p = \frac{p - p^*}{q^*} = - 2\phi_x + \phi_y^2 \qquad (14)$$

where q^* is the sonic dynamic pressure. The contribution of the ϕ_y^2 to the C_p in the present case for $\theta_0 = 1/10$ amounted to about 10% of the total. The result for the cone with the nose angle of 10° also shown in Figure 7 is found by the similarity law which is discussed in the next section. Figure 8 shows the pressure distribution over both the conical nose and the cylindrical afterbody for the case that the nose angle equals 10°. The pressures on the cylindrical afterbody were computed by the method of characteristics.

SECTION VII

VERIFICATION OF THE TRANSONIC SIMILARITY LAW

Von Karman (Reference 2) and Oswatitsch and Berndt (Reference 5) have proposed a transonic similarity law for axial symmetric bodies, but because of the slender body approximation used in the boundary condition at the body there is some question of the validity of this law for bodies of practical slenderness ratios. The results of the present paper may be used to check the validity of their similarity law.

In order to see how the slender body theory was introduced let us rederive the similarity law for the special case of the cone-cylinder body at Mach number one. The starting point is the boundary value problem formulated in Section IV with however the upstream and upper boundary conditions being replaced by the original conditions at infinity. We consider therefore the problem of seeking a solution of the equation

$$-(\kappa+1)\,\phi_x\,\phi_{xx} + \phi_{yy} + \frac{1}{y}\,\phi_y = 0 \tag{15}$$

with the boundary conditions:

1. along the surface of the cone, $y = \theta_0\, x$ (here the origin of the x is taken at the nose of the cone): $\phi_y = \theta_0$,

2. along the axis of symmetry upstream of the cone: $\phi_y = 0$,

3. at infinity: $\phi_x = 0$, $\phi_y = 0$, and

4. at the shoulder of the cone the flow is locally a Meyer expansion.

The essential idea of a similarity law is to obtain from a known solution over a given body that over another body (generally affine to the first) by a transformation. The procedure is to introduce a scale transformation to the dependent and independent variables and attempt to transform the above boundary value problem such that the parameter θ_0 characterizing the slenderness ratio of the cone does not appear explicitly in the transformed problem. If this is possible, then one can solve the transformed boundary value problem and then obtain the flow over any cone-cylinder body with the semi-nose θ_0 from this solution by

the given transformation.

Consider now the scale transformation

$$\bar{x} = x$$

$$\bar{y} = c\, y$$

$$c'\, \bar{\phi}(\bar{x},\bar{y}) = \phi(x,y)$$

where c and c' are constants. Inserting this into the boundary value problem it is found that the differential equation for $\bar{\phi}$ is the same as for ϕ if one chooses $c' = c^2$; that is, the differential equation is invariant under the scale transformation if c and c' are related in this way. The boundary conditions at infinity as well as the condition of symmetry along the x-axis will also be invariant under the transformation, and will therefore not depend upon θ_0. There is however no possibility to eliminate the θ_0 from the boundary condition at the cone surface. At this point the slender body approximation has been introduced by von Karman and Oswatitsch and Berndt to avoid this difficulty.

In this approximation the differential equation (15) is first simplified by assuming that the body on hand is sufficiently slender that the variations of ϕ in the x-direction in the vicinity of the body are negligible as compared to those in the y-direction. The differential equation may be then simplified in the region near the body to $\phi_{yy} + \frac{1}{y}\phi_y = 0$. This equation can of course be integrated with the result $\phi = f_1 \log y + f_2$ where f_1 and f_2 are functions of x alone. In particular it is seen from this solution that the quantity $y\phi_y$ is independent of y. This fact is then used to reformulate the boundary value problem such that for the boundary condition along the body the quantity $y\phi_y = \theta_0^2\, x$ is prescribed instead of $\phi_y = \theta_0$. Since for slender bodies $y\phi_y$ does not depend on y, it may be prescribed along any line in the vicinity of the body. Let us prescribe it along the line $y = 0$. Carrying out the transformation (16) in this condition with $c' = c^2$, one obtains

$$\bar{y}\, \bar{\phi}_{\bar{y}} = \frac{\theta_0^2}{c^2}\, \bar{x}$$

If now one puts $c = \theta_0$, then this boundary condition becomes also independent of the θ_0 that is, one obtains the condition $\bar{y}\, \bar{\phi}_{\bar{y}} = \bar{x}$ for $\bar{y} = 0$ and $\bar{x}$ ranging from 0 to the value of $\bar{x}$ at the shoulder.

With regard to the boundary condition at the shoulder (that concerning the Meyer expansion) one cannot expect the slender body

approximation to hold for this condition. This will be of little consequence, however, with regard to the subsonic flow since it was seen from the computations that the influence of this boundary condition was confined only to that portion of the subsonic flow in the vicinity of the sonic line, and that there was negligible effect on the pressure distribution over the cone. Consequently from the point of view of the subsonic flow one obtains a boundary value problem for $\bar{\phi}$ which does not depend upon the parameter θ_0.

From $\bar{\phi}$ one obtains the solution ϕ for the flow over a given cone-cylinder body with the semi-nose angle of θ_0 by the transformation (16) with $c' = c^2 = \theta_0^2$. The pressure coefficient C_p along this cone is then found from the equation

$$C_p(x, y_b; \theta_0) = -2\phi_x(x, y_b, \theta_0) + \theta_0^2 = -2\theta_0^2 \bar{\phi}_{\bar{x}}(\bar{x}, \bar{y}_b) + \theta_0^2 \quad (17)$$

where $y_b = \theta_0 x$ and $\bar{y}_b = \theta_0^2 \bar{x}$; here $\bar{y} = \bar{y}_b$ is the line corresponding to $y = y_b$.

There is a difference between the results of von Karman and that of Oswatitsch and Berndt with regard to the variation of the pressure coefficient with θ_0. Aside from the fact that von Karman did not include the ϕ_y^2 term for the C_p it appears that in his result the value of $\bar{\phi}_{\bar{x}}$ was not evaluated along $\bar{y} = \bar{y}_b$ as was done by Oswatitsch and Berndt.

Using Equation (17) one can now find the relationship between the pressure coefficients C_{p_1} and C_{p2} at the surface of two cone-cylinder bodies of semi-nose angles θ_{0_1} and θ_{02} respectively; that is

$$C_{p_2}(x) = \left(\frac{\theta_{0_2}}{\theta_{0_1}}\right)^2 \left(C_{p_1}(x) - 2\theta_{0_1}^2 \ln\frac{\theta_{0_2}^2}{\theta_{0_1}^2} \right) \quad (18)$$

This result is a specialization of the result of Oswatitsch and Berndt (Reference 5).

It is seen that the crucial point of the above similarity law was the possibility of fulfilling the modified boundary condition for the body along the axis of symmetry; that is, it was essential that the quantity $y\phi_y = \phi_\eta$ was independent of y in the vicinity of the body. It is seen in Figure 5 that for the cone-cylinder body of $\theta_0 = 1/10$ this requirement is indeed fulfilled for a considerable distance from the body and for all positions along the cone with the exception of the region near the nose.

It can therefore be concluded that the transonic similarity law can be extended to cover cone-cylinder bodies of practical slenderness ratios.

SECTION VIII

CONCLUDING REMARKS

It was found that the numerical procedure proposed in the present paper appeared to give a definite solution to the boundary value problem with however a tremendous amount of computational work. It is not clear whether the slow convergence of the present problem was due solely to the type of the differential equation: it is suspected that the nature of the boundary conditions played an essential role.

The computations indicated that the pressure distribution over the body was practically independent of the supersonic boundary condition, or in other words, the location of the sonic line. The same results could have been obtained by assuming a straight sonic line—a condition which is of course physically impossible. It is not clear what effect the remaining residuals have upon the flow as a whole, but it was fairly evident that they would have little influence upon the pressure distribution over the cone.

The results of the computations showed that the slender body approximation used by von Karman and Oswatitsch and Berndt in their derivation of the transonic similarity law is valid for the present example, and that this law can be applied to bodies of practical slenderness ratios. The pressure distributions over similar bodies are then found from Equation (18) due to Oswatitsch and Berndt.

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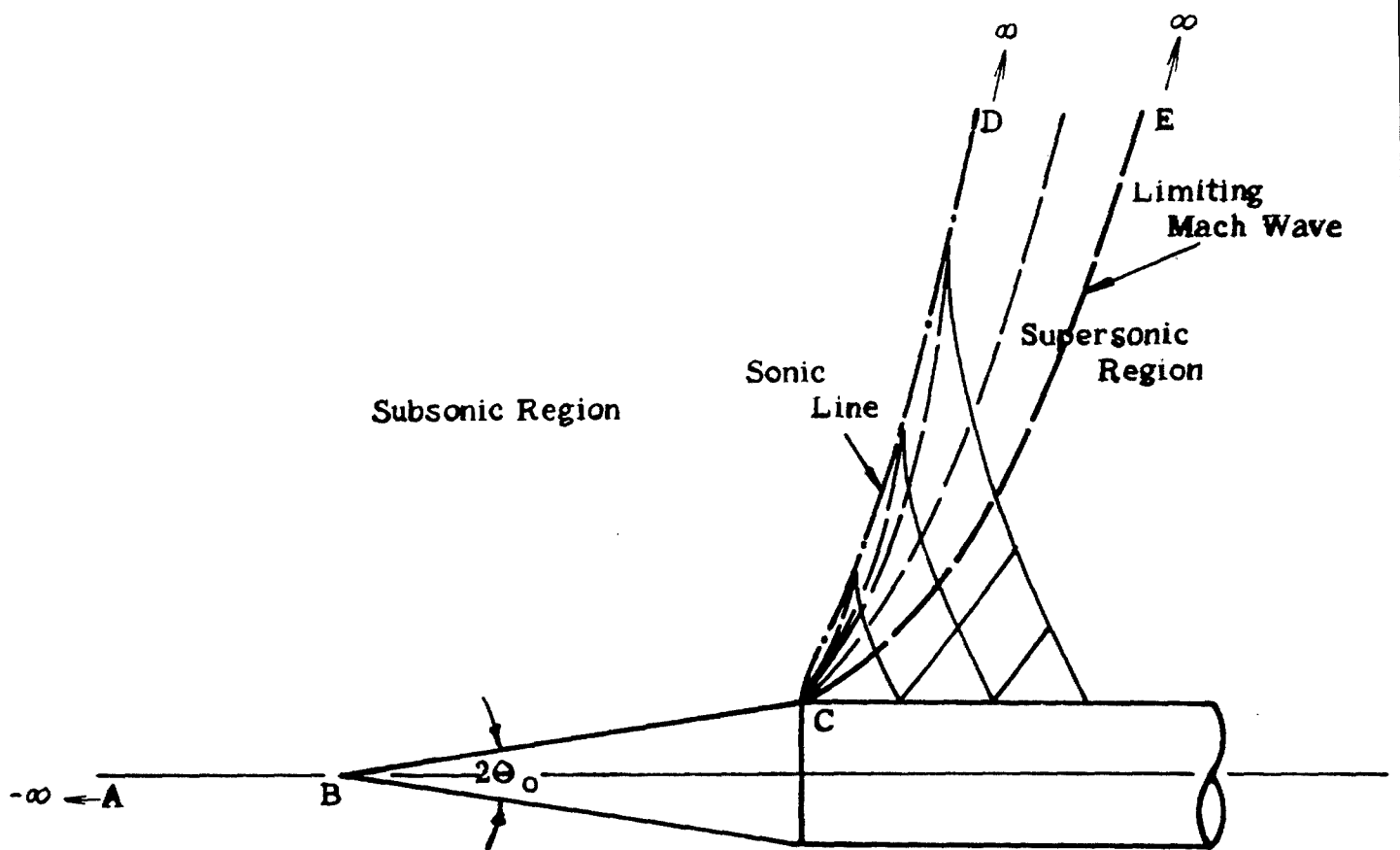
Figure 1: Flow over a Cone-Cylinder Body
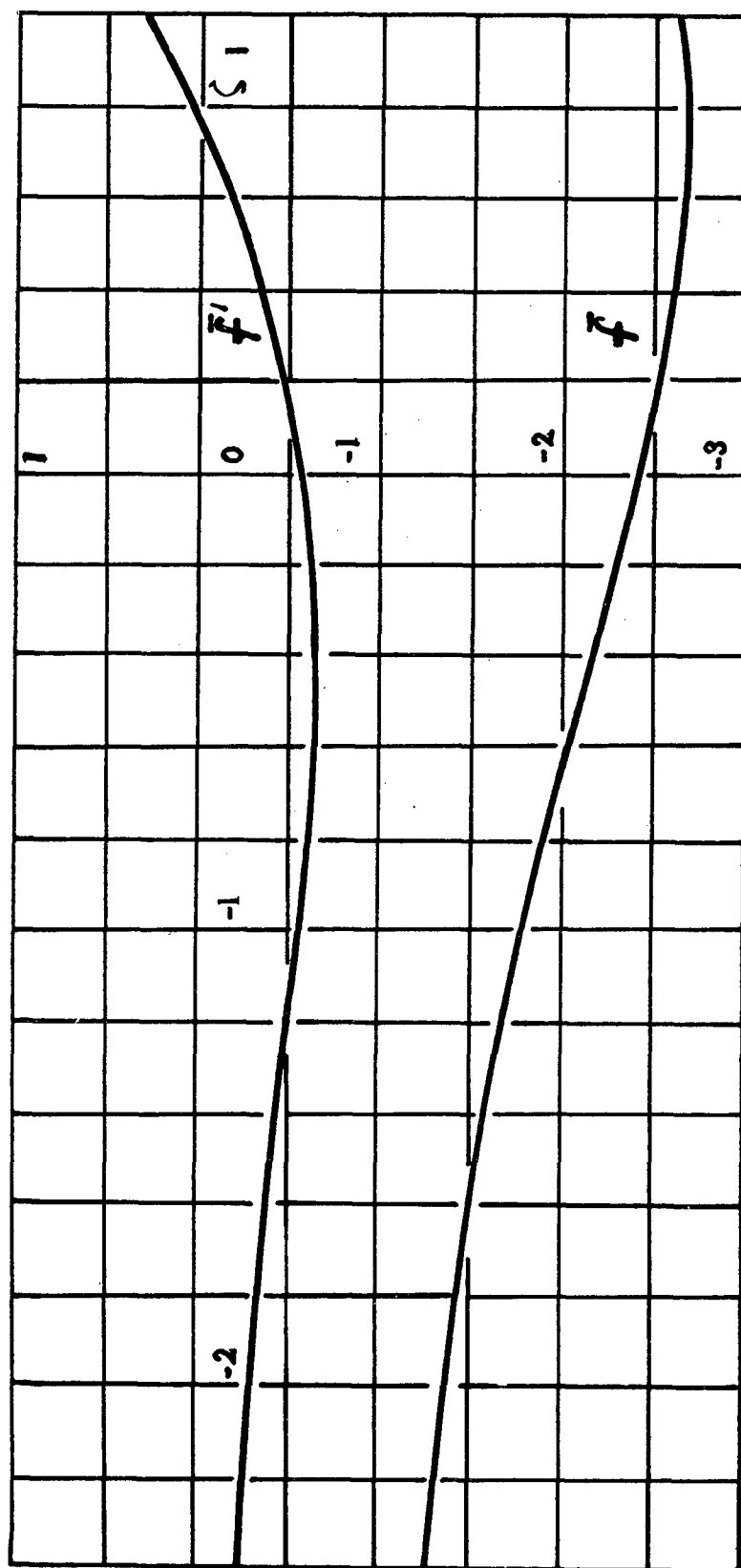
Figure 2: Asymptotic Solution of Reference 4.
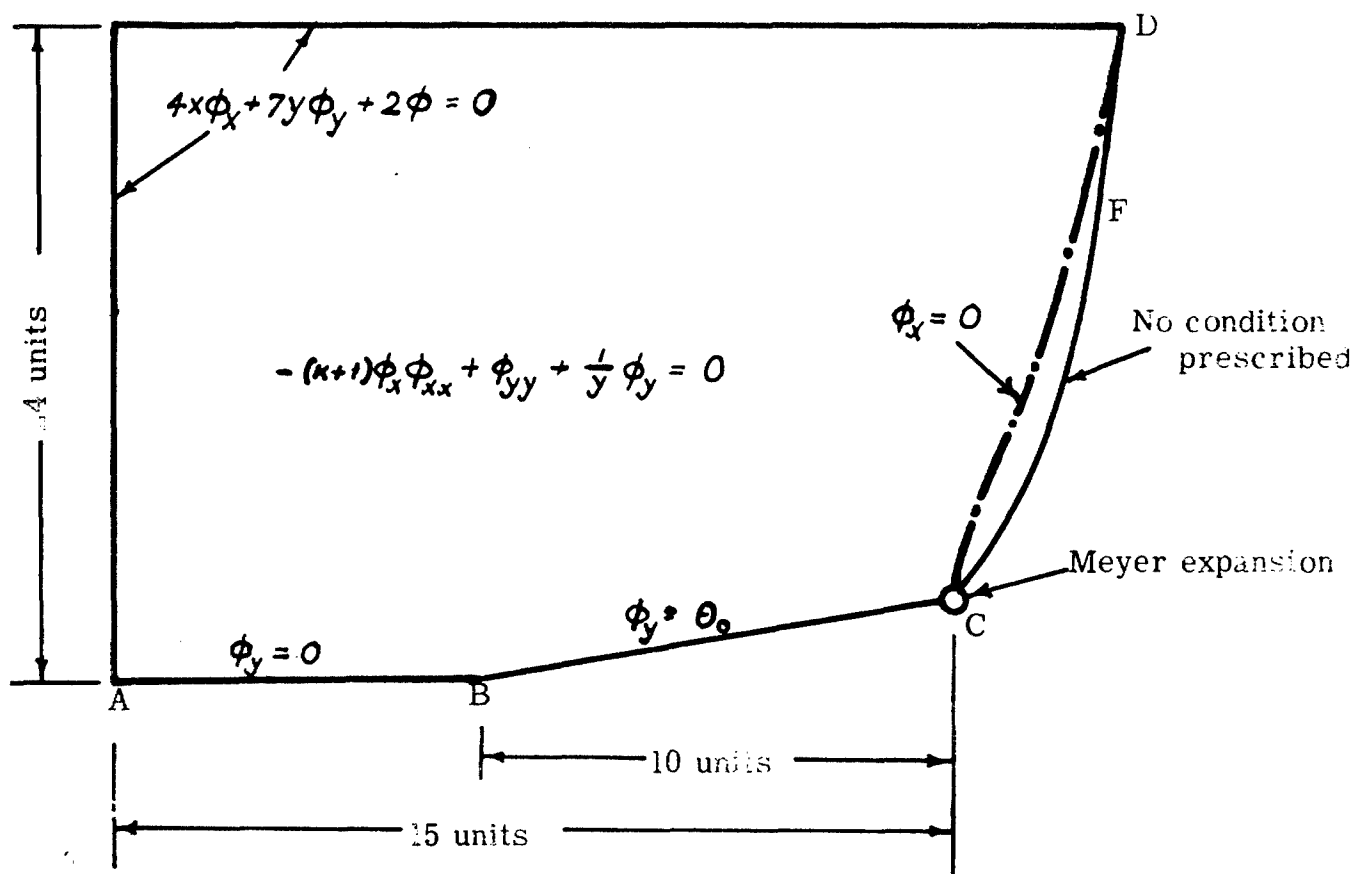
Figure 3: Boundary Conditions for the Cone-Cylinder Body
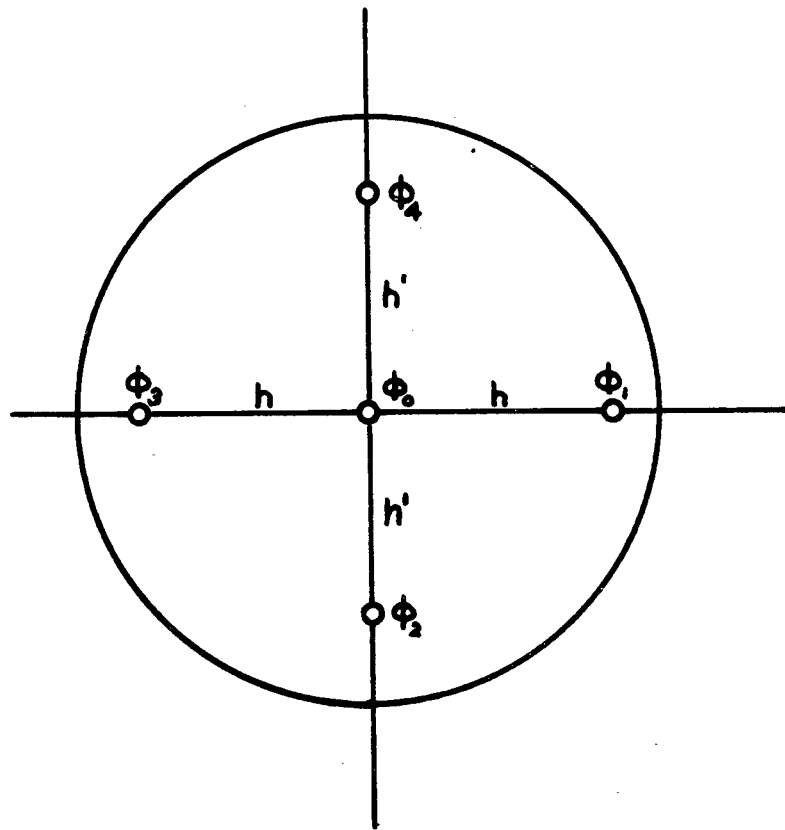
FIGURE 4: A TYPICAL SUBDOMAIN

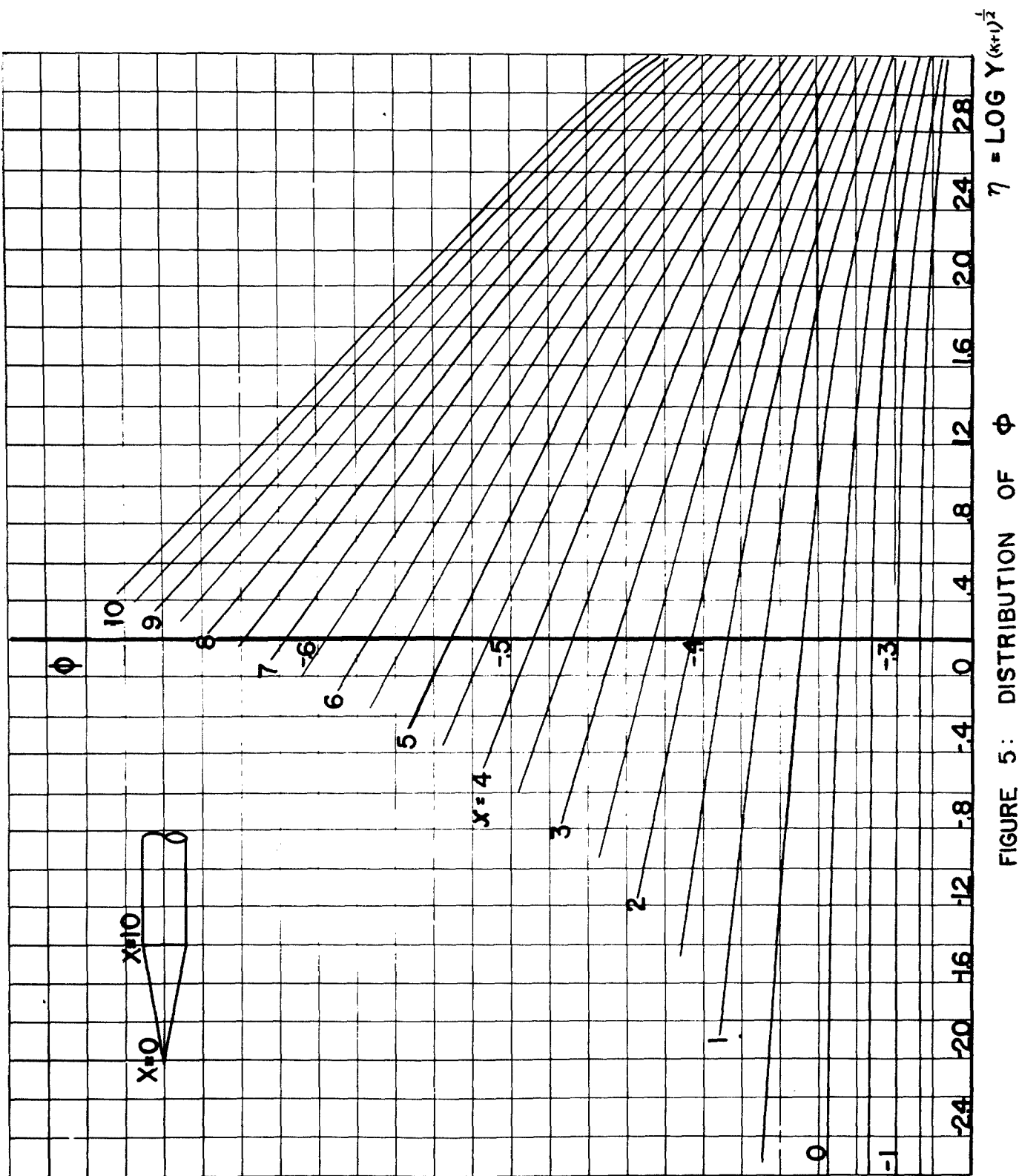FIGURE 5: DISTRIBUTION OF Φ

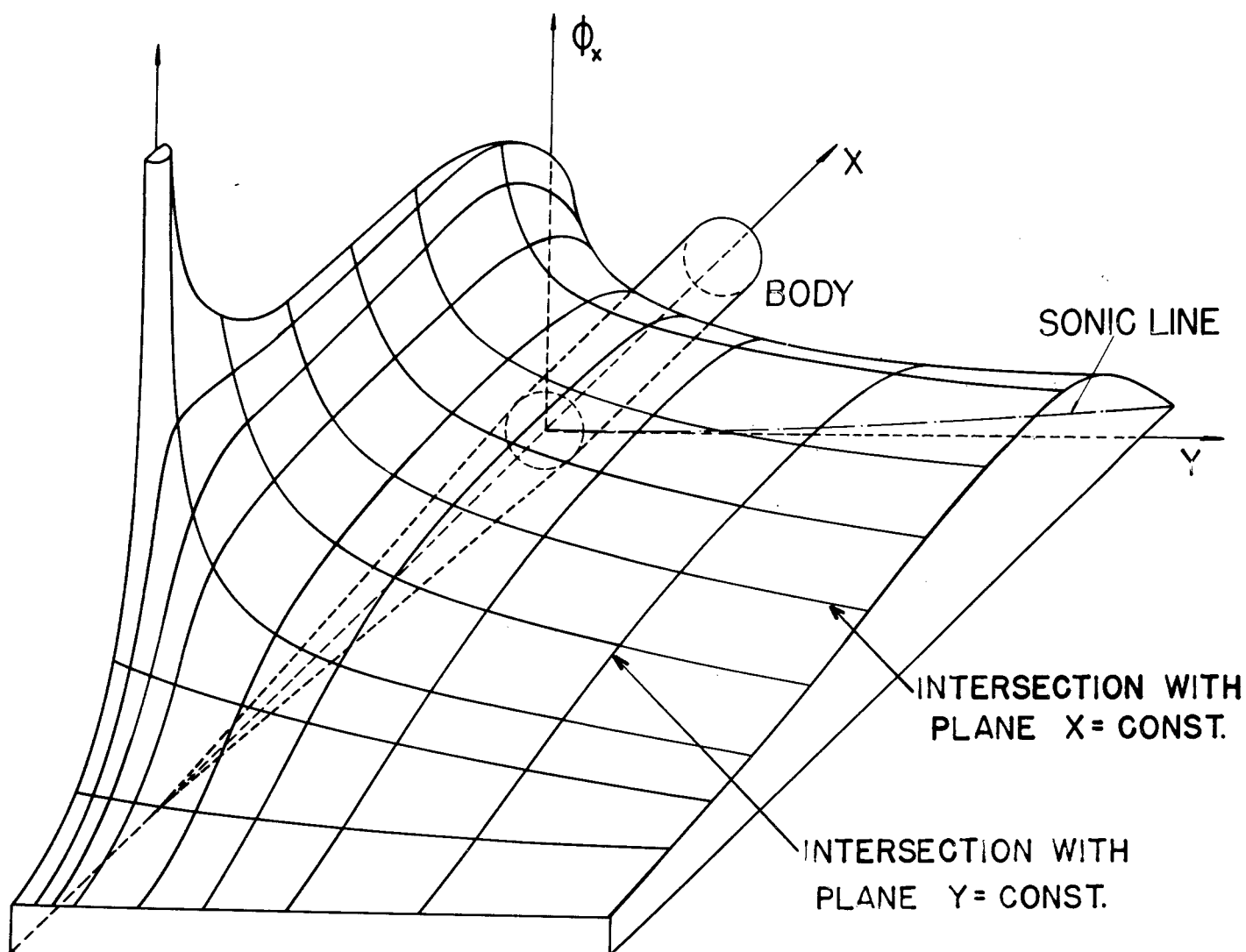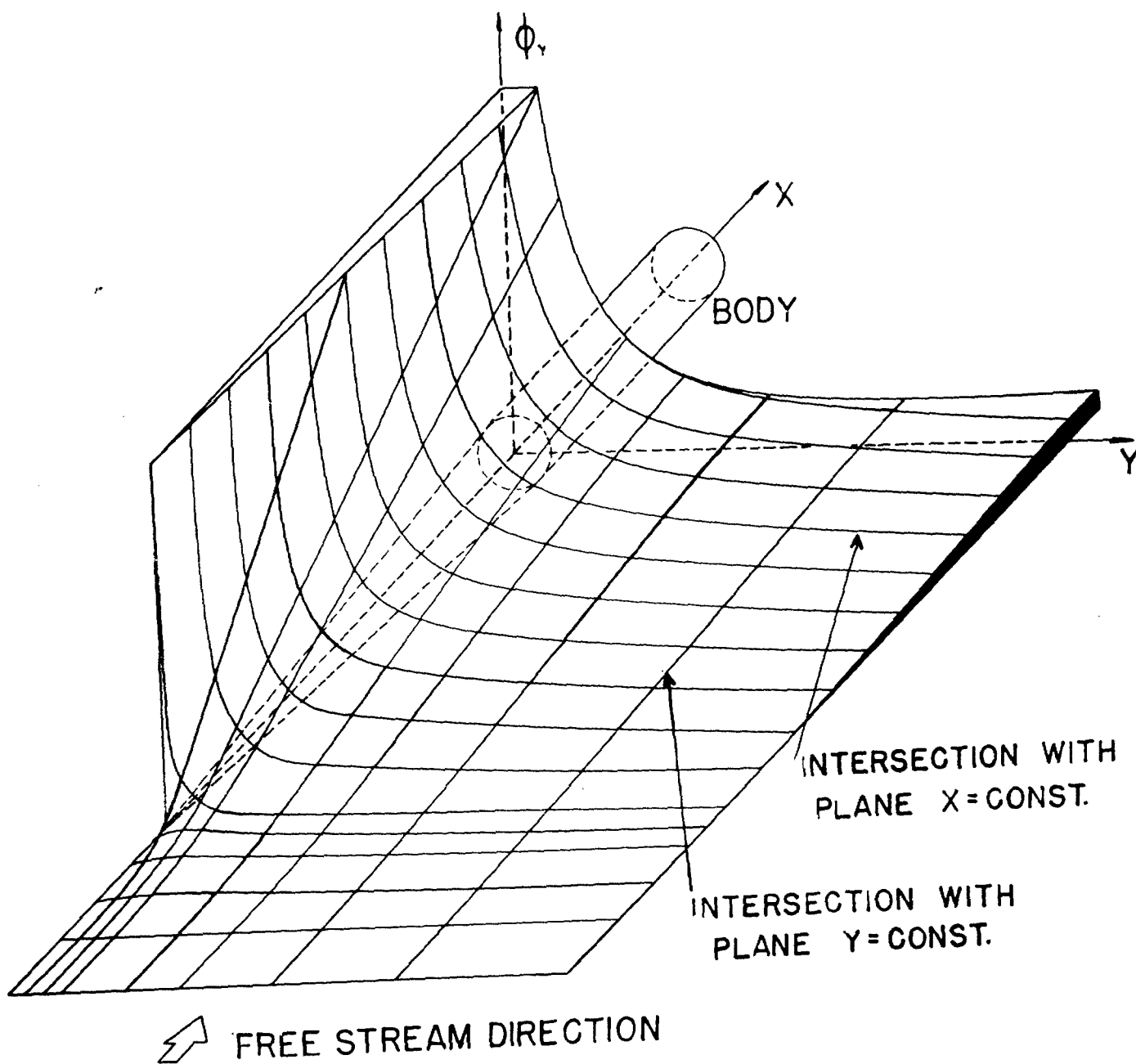FIGURE 6 a: DISTRIBUTION OF ϕ_x

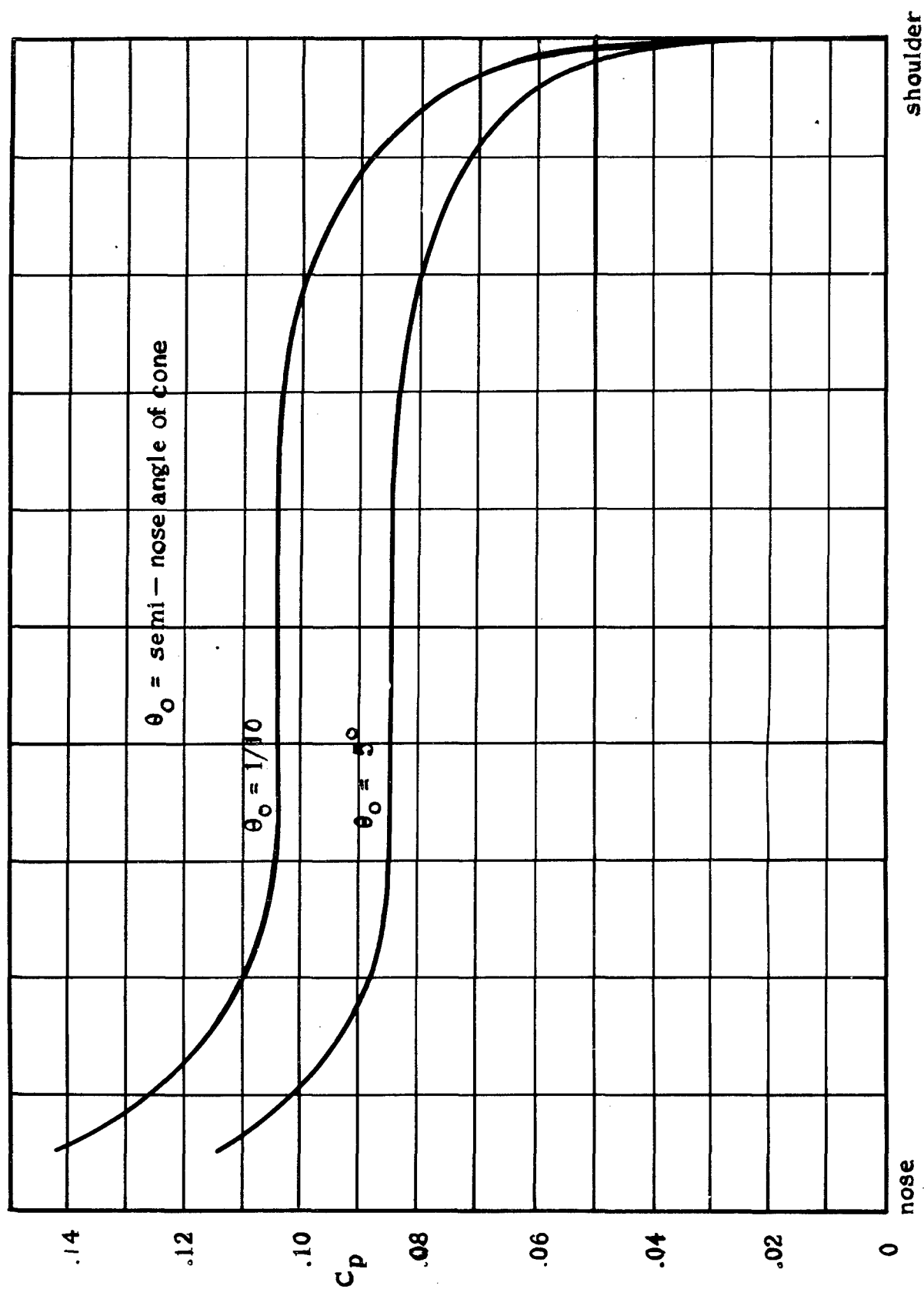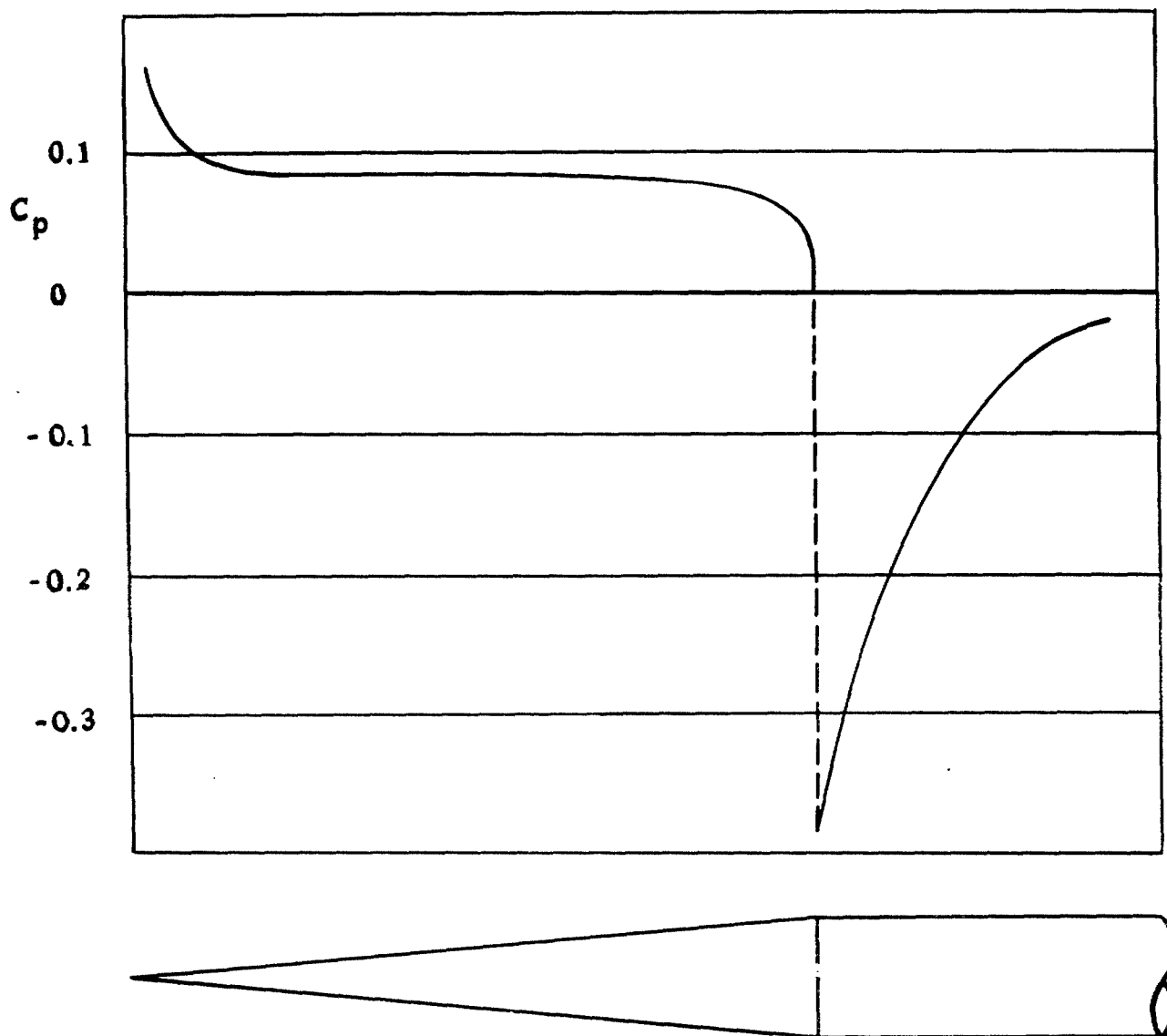FIGURE 6b: DISTRIBUTION OF ϕ_y

FIGURE 7: PRESSURE DISTRIBUTION ON THE CONICAL NOSE

FIGURE 8: PRESSURE DISTRIBUTION OVER A CONE - CYLINDER BODY
WITH A NOSE ANGLE OF 10°